Consequences of the H theorem from nonlinear Fokker-Planck equations

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A general type of nonlinear Fokker-Planck equation is derived directly from a master equation, by introducing generalized transition rates. The H theorem is demonstrated for systems that follow those classes of nonlinear Fokker-Planck equations, in the presence of an external potential. For that, a relation involving terms of Fokker-Planck equations and general entropic forms is proposed. It is shown that, at equilibrium, this relation is equivalent to the maximum-entropy principle. Families of Fokker-Planck equations may be related to a single type of entropy, and so, the correspondence between well-known entropic forms and their associated Fokker-Planck equations is explored. It is shown that the Boltzmann-Gibbs entropy, apart from its connection with the standard—linear Fokker-Planck equation—may be also related to a family of nonlinear Fokker-Planck equations.

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I. INTRODUCTION

It is well known that many real systems exhibit a dynamical behavior that falls out of the scope of the standard linear differential equations of physics. Although the linear Fokker-Planck equation (FPE) [1] is considered appropriate for the description of a wide variety of physical phenomena—typically those associated with normal diffusion—it is well accepted that this equation is not adequate for describing anomalous diffusion. An example consists of particle transport in disordered media [2], such as amorphous materials, or some other kind of media containing impurities and/or defects. In such systems, particles are driven by highly irregular forces, which lead to transport coefficients that may vary locally in a nontrivial manner. In addition to those, various other phenomena fall out of the scope of the linear FPE, such as surface growth [3], diffusion of micelles in salted water [4], polytropic distributions of self-gravitating stellar systems [5], the heartbeat histograms in a healthy individual [6], relaxation of two-dimensional turbulence in a pure-electron plasma [7], anomalous diffusion in an optical lattice [8], and black-hole radiation [9]. These kinds of phenomena became one of the most investigated topics in physics nowadays, leading to many interesting new aspects and to a wide range of open problems.

In order to cope with such anomalous systems, modifications in the linear FPE have been carried out, and this subject has attracted the attention of many researchers recently. Essentially, there are two alternatives for introducing modifications in the linear FPE: (i) a procedure that leads to the fractional FPE (see Ref. [10] for a review), where one considers a linear theory with nonlocal operators carrying the anomalous nature of the process; (ii) the nonlinear FPEs [11] that in most of the cases come out as simple phenomenological generalizations of the standard linear FPE [12–20]. Recently, it has been shown that nonlinear FPEs may be derived directly from a standard master equation, by introducing nonlinear effects on its associated transition probabilities [21–23].

The nonextensive statistical mechanics formalism has emerged naturally as a strong candidate for dealing appropriately with many real systems that are not satisfactorily described within standard (extensive) statistical mechanics [24–26]. The powerlike probability distribution that maximizes the entropy proposed by Tsallis [27–29] is very often found as a solution of nonlinear FPEs [12–16,18], suggesting that the nonextensive statistical mechanics formalism should be intimately related to nonlinear FPEs. In what concerns thermodynamics, one requires the extensivity of the entropy; however, there are several examples in the literature for which the Boltzmann-Gibbs entropy is not extensive [26,30], including quantum spin systems [31,32]. This suggests the utility of alternative entropic forms that may be, in certain cases, extensive.

Many important equations and properties of standard statistical mechanics have been extended within the formalism of nonextensive statistical mechanics. An example is the H theorem, which was shown to be valid, taking into account certain restrictions on the parameters of the corresponding entropic form [11,18,33–35]. Usually, one proves the H theorem by defining previously an entropic form, and then considering either the master equation or a FPE, when dealing with the time derivative of the probability distribution.

The main motivation of this paper is to prove the H theorem for a system in the presence of an external potential and following a general type of nonlinear FPE. In order to achieve this, we introduce a relation involving quantities of the FPE with an entropic form; in principle, one may have classes of Fokker-Planck equations associated with a single entropic form. We show that, when considered at equilibrium, this relation is equivalent to the maximum-entropy principle. In the next section we derive a general FPE directly from a master equation by introducing nonlinear terms in its transition probabilities; such a FPE will be used throughout most of this paper. In Sec. III we prove the H theorem by using this FPE, and show that the validity of this theorem requires a relation involving a general entropic form and the parameters of this nonlinear FPE. In Sec. IV we discuss particular cases of this FPE and their associated entropic forms. In Sec. V we introduce a modified FPE that is

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compatible with the definition of a "generalized internal energy," as used within the context of nonextensive statistical mechanics. The same relation introduced previously is also necessary in this case, in order to prove the H theorem. Finally, in Sec. VI we present our conclusions.

II. DERIVATION OF THE NONLINEAR FOKKER-PLANCK EQUATION FROM A MASTER EQUATION

In this section we will derive, directly from the master equation, the nonlinear FPE that will be investigated throughout most of the present paper; we will follow closely the approach used in Refs. [21,22]. Let us then consider the standard master equation, associated with a discrete spectrum,

$$\frac{\partial P(n,t)}{\partial t} = \sum_{m=-\infty}^{\infty} [P(m,t)w_{m,n}(t) - P(n,t)w_{n,m}(t)], \quad (2.1)$$

with $P(n,t)$ representing the probability for finding a given system in a state characterized by a variable n, at time t. We introduce nonlinearities in the system through the following transition rates:

$$w_{k,l}(\Delta) = -\frac{1}{\Delta}\delta_{k,l+1}A(k\Delta)\ a[P(k\Delta,t)] + \frac{1}{\Delta^2}(\delta_{k,l+1} + \delta_{k,l-1})\ \Upsilon[P(k\Delta,t),R(l\Delta,t)]. \quad (2.2)$$

In the equation above, $A(k\Delta)$ represents an external dimensionless force, $a[P]$ is a functional of the probability $P(n,t)$, whereas the functional $\Upsilon[P,R]$ depends on two probabilities, P and R, that are associated with two different states, although $R(k\Delta,t) \equiv P(k\Delta,t)$. Substituting this transition rate in Eq. (2.1), performing the sums, and defining $x=k\Delta$, one gets

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} = & -\frac{1}{\Delta}\{P(x+\Delta,t)A(x+\Delta)a[P(x+\Delta,t)] \\ & - P(x,t)A(x)a[P(x,t)]\} \\ & + \frac{1}{\Delta^2}\{P(x+\Delta,t)\Upsilon[P(x+\Delta,t),R(x,t)] \\ & + P(x-\Delta,t)\Upsilon[P(x-\Delta,t),R(x,t)]\} - \frac{1}{\Delta^2}P(x,t) \\ & \times\{\Upsilon[P(x,t),R(x+\Delta,t)] + \Upsilon[P(x,t),R(x-\Delta,t)]\}.\end{aligned} \quad (2.3)$$

The quantities depending on Δ may be expanded for small Δ, e.g.,

$$\begin{aligned}\Upsilon[P(x,t),R(x\pm\Delta,t)] = \Bigg[& \Upsilon[P(x,t),R(x,t)] + \left(\pm\Delta\frac{\partial R(x,t)}{\partial x} + \frac{\Delta^2}{2}\frac{\partial^2 R(x,t)}{\partial x^2}\right)\frac{\partial \Upsilon[P,R]}{\partial R} \\ & + \frac{\Delta^2}{2}\left(\frac{\partial R(x,t)}{\partial x}\right)^2\frac{\partial^2\Upsilon[P,R]}{\partial R^2} + \cdots\Bigg]_{R=P},\end{aligned} \quad (2.4)$$

in such a way that considering the limit $\Delta \to 0$, one gets the nonlinear FPE,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}, \quad (2.5)$$

with

$$\Psi[P(x,t)] = P(x,t)a[P(x,t)], \quad (2.6a)$$

$$\Omega[P(x,t)] = \left[\Upsilon[P,R] + P(x,t)\left(\frac{\partial\Upsilon[P,R]}{\partial P} - \frac{\partial \Upsilon[P,R]}{\partial R}\right)\right]_{R=P}, \quad (2.6b)$$

where we have used the fact that $\partial P(x,t)/\partial x \equiv \partial R(x,t)/\partial x$. The external force $A(x)$ is associated with a potential $\phi(x)$ $[A(x)=-d\phi(x)/dx,\ \phi(x)=-\int_{-\infty}^{x}A(x')dx']$, and we are assuming analyticity of the potential $\phi(x)$, as well as integrability of the force $A(x)$ in all space. Furthermore, the functionals $\Psi[P(x,t)]$ and $\Omega[P(x,t)]$ are supposed to be both positive finite quantities, integrable as well as differentiable (at least once) with respect to the probability distribution $P(x,t)$; i.e., they should be at least, $\Omega[P],\Psi[P] \in C^1$. In addition to that, $\Psi[P(x,t)]$ should be also a monotonically increasing functional of $P(x,t)$.

As usual, we assume that the probability distribution, together with its first derivative, as well as the product $A(x)\Psi[P(x,t)]$, should all be zero at infinity,

$$P(x,t)|_{x\to\pm\infty} = 0, \qquad \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0, \qquad A(x)\Psi[P(x,t)]|_{x\to\pm\infty} = 0 \quad (\forall\ \ t). \quad (2.7)$$

The conditions above guarantee the preservation of the normalization for the probability distribution, i.e., if for a given time t_0 one has that $\int_{-\infty}^{\infty}dx\ P(x,t_0)=1$, then a simple integration of Eq. (2.5) with respect to the variable x yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}dx\ P(x,t) = -\{A(x)\Psi[P(x,t)]\}_{-\infty}^{\infty} + \left(\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right)_{-\infty}^{\infty} = 0, \quad (2.8)$$

and so,

$$\int_{-\infty}^{\infty}dx\ P(x,t) = \int_{-\infty}^{\infty}dx\ P(x,t_0) = 1 \quad (\forall\ \ t). \quad (2.9)$$

It is important to stress that the nonlinear FPE of Eq. (2.5) is very general and reproduces well-known FPEs of the literature. As examples, one should mention the particular cases: (i) the linear FPE is recovered for $\Psi[P(x,t)]=P(x,t)$ and $\Omega=D$ (constant); (ii) the nonlinear FPE that presents Tsallis distribution as a solution [12,13], is obtained by setting $\Psi[P(x,t)]=P(x,t)$ and $\Omega[P(x,t)]=qD[P(x,t)]^{q-1}$, where q is the well-known entropic index [27], characteristic of the nonextensive statistical mechanics formalism; (iii) the nonlinear FPE derived previously from the master equation [21–23] is recovered for $\Psi[P(x,t)]=P(x,t)$.

From the experimental point of view, particular cases of the nonlinear FPE of Eq. (2.5) were shown to be important for the description of several physical phenomena: (i) particle transport in porous media $[A(x)=0,\ \Omega[P(x,t)]=D[P(x,t)]^{\mu},\ \mu\in\Re]$ [2] and a specific application of this equation corresponds to a diffusion of micelles in salted water, in which case $\mu=-3/2$ [4]; (ii) surface growth, which may be governed by nonlinear equations characterized by complicated force and diffusive terms [3].

In the next section we prove the H theorem for a system in the presence of an external potential and following the general type of nonlinear FPE of Eq. (2.5).

III. H THEOREM

Herein, we will consider a general type of entropic form, satisfying the following conditions:

$$S[P]=\Lambda(Q[P]),\quad Q[P]=\int_{-\infty}^{\infty}dx\ g[P(x,t)],$$

$$g(0)=g(1)=0,\quad \frac{d^2g}{dP^2}\le 0, \tag{3.1}$$

where $\Lambda[Q]$ represents a monotonically increasing outer functional with dimensions of entropy that is supposed to satisfy, at least, $\Lambda[Q]\in C^1$, whereas the inner functional $g[P(x,t)]$ should be also, at least, $g[P(x,t)]\in C^2$ in the interval $0<P(x,t)<1$ (end points excluded). Since we are dealing with a system that exchanges energy with its surrounding, herein represented by the potential $\phi(x)$, it is important to define also the free-energy functional,

$$F=U-\frac{1}{\beta}S,\quad U=\int_{-\infty}^{\infty}dx\ \phi(x)P(x,t), \tag{3.2}$$

where β represents a positive Lagrange multiplier.

The H theorem, for a system subject to an external potential, corresponds to a well-defined sign for the time derivative of the above free-energy functional, which we will consider as $(dF/dt)\le 0$. Using the definitions above,

$$\frac{dF}{dt}=\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty}dx\ \phi(x)P(x,t)-\frac{1}{\beta}\Lambda(Q[P])\right)$$
$$=\int_{-\infty}^{\infty}dx\left(\phi(x)-\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right)\frac{\partial P}{\partial t}, \tag{3.3}$$

where we remind one that $\Lambda[Q]$ and $d\Lambda[Q]/dQ$ do not depend on the variable x. Now, one may use the FPE of Eq. (2.5) for the time derivative of the probability distribution; carrying an integration by parts, and assuming the conditions of Eq. (2.7), one gets

$$\frac{dF}{dt}=-\int_{-\infty}^{\infty}dx\left(\frac{d\phi(x)}{dx}\Psi[P]+\Omega[P]\frac{\partial P}{\partial x}\right)$$
$$\times\left(\frac{d\phi(x)}{dx}-\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2}\frac{\partial P}{\partial x}\right). \tag{3.4}$$

In most of the cases, one is interested in verifying the H theorem by using a well-defined FPE, together with particular entropic forms, in such a way that some of the quantities, $\Lambda[Q]$, $\Omega[P]$, $\Psi[P]$, and $d^2g[P]/dP^2$, are previously defined (see, e.g., Refs. [18,35]). Herein, we follow a more general approach, i.e., we assume that Eqs. (2.5), (2.7), (3.1), and (3.2) are satisfied, and then, we impose the condition

$$-\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2}=\frac{\Omega[P]}{\Psi[P]}. \tag{3.5}$$

Using this condition, Eq. (3.4) may be written as

$$\frac{dF}{dt}=-\int_{-\infty}^{\infty}dx\ \Psi[P]\left(\frac{d\phi(x)}{dx}+\frac{\Omega[P]}{\Psi[P]}\frac{\partial P}{\partial x}\right)^2\le 0, \tag{3.6}$$

and we remind one that $\Psi[P]$ is a positive, monotonically increasing functional of $P(x,t)$.

It should be stressed that Eq. (3.5) expresses an important relation involving quantities of the FPE and possible entropic forms, for the case of a system in the presence of an external potential. It leads to a correspondence between whole families of FPEs, defined in terms of the functionals $\Omega[P]$ and $\Psi[P]$, with a single entropic form. Therefore, it allows the calculation of the entropic form associated with a given class of FPEs; on the other hand, one may also start by considering a given entropic form and then find the class of FPEs associated with it. In fact, since the FPE is a phenomenological equation that specifies the dynamical evolution associated with a given physical system, Eq. (3.5) may be useful in the identification of the entropic form associated with such a system. In particular, one may identify entropic forms associated with some anomalous systems, exhibiting unusual behavior that are appropriately described by nonlinear FPEs, like the one of Eq. (2.5). Within the present approach, the relation of Eq. (3.5) should hold for the H theorem to be valid; even though the relation of Eq. (3.5) may not be unique, we shall argue its relevance in what follows.

First of all, let us show that at equilibrium, Eq. (3.5) is equivalent to the maximum-entropy principle. For that, we introduce the functional

$$\mathcal{I}[P(x,t)]=\Lambda(Q[P])+\alpha\left(1-\int_{-\infty}^{\infty}dx\ P(x,t)\right)$$
$$+\beta\left(U-\int_{-\infty}^{\infty}dx\ \phi(x)P(x,t)\right), \tag{3.7}$$

where α and β are Lagrange multipliers. Then, one has that

$$\left.\frac{d\mathcal{I}[P]}{dP}\right|_{P=P_{\text{eq}}(x)} = 0 \Rightarrow \left.\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right|_{P=P_{\text{eq}}(x)} - \alpha - \beta\phi(x) = 0, \tag{3.8}$$

where $P_{\text{eq}}(x)$ represents the equilibrium probability distribution.

From the general FPE of Eq. (2.5), one gets that, at equilibrium,

$$A(x) = \frac{\Omega[P_{\text{eq}}]}{\Psi[P_{\text{eq}}]}\frac{dP_{\text{eq}}(x)}{dx}, \tag{3.9}$$

which, after integration, yields

$$\phi_0 - \phi(x) = \int_{x_0}^{x} dx \frac{\Omega[P_{\text{eq}}]}{\Psi[P_{\text{eq}}]}\frac{dP_{\text{eq}}(x)}{dx} = \int_{P_{\text{eq}}(x_0)}^{P_{\text{eq}}(x)} \frac{\Omega[P_{\text{eq}}(x')]}{\Psi[P_{\text{eq}}(x')]} dP_{\text{eq}}(x'), \tag{3.10}$$

where $\phi_0 \equiv \phi(x_0)$ is a constant. Integrating Eq. (3.5), at equilibrium,

$$\left.\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right|_{P_{\text{eq}}(x)} = \phi(x) + C_1, \tag{3.11}$$

where we have used Eq. (3.10), and C_1 is a constant resulting from the above integration. One notices that the equation above is equivalent to the one obtained from the maximum-entropy principle [cf. Eq. (3.8)].

An important—and complementary—property required for a functional satisfying the H theorem is that it should be bounded from below,

$$F(P(x,t)) \geq F(P_{\text{eq}}(x)) \quad (\forall \ t). \tag{3.12}$$

Herein, we assume the presence of a unique equilibrium state in the functional $F(P(x,t))$. In this case, Eq. (3.12) together with the imposition from the H theorem, for a time-decreasing functional F, ensure that, after a long time, the system will always reach equilibrium. Therefore, it is sufficient to prove that the requirement of Eq. (3.12) holds only in the nearness of the global equilibrium. Let us then consider,

$$F(P) - F(P_{\text{eq}}) = \int_{-\infty}^{\infty} dx\ \phi(x)(P - P_{\text{eq}}) - \frac{1}{\beta}\{\Lambda(Q[P]) - \Lambda(Q[P_{\text{eq}}])\}, \tag{3.13}$$

which may be expanded, near the equilibrium, up to $O[(P-P_{\text{eq}})^2]$. It should be noticed that an expansion on the probability $P(x,t)$, near $P_{\text{eq}}(x)$, implies an expansion of the functional $\Lambda(Q[P])$ in powers of $Q[P]-Q[P_{\text{eq}}]$; carrying out such an expansion, one gets that

$$F(P) - F(P_{\text{eq}}) = \int_{-\infty}^{\infty} dx\left\{(P-P_{\text{eq}})\left(\phi(x) - \left.\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right|_{P_{\text{eq}}(x)}\right) + \frac{1}{2}(P-P_{\text{eq}})^2\left(-\left.\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2}\right|_{P_{\text{eq}}(x)}\right)\right\} - \left.\frac{1}{2\beta}\frac{d^2\Lambda[Q]}{dQ^2}\right|_{P_{\text{eq}}(x)}\left(\int_{-\infty}^{\infty} dx \left.\frac{dg[P]}{dP}\right|_{P_{\text{eq}}(x)}(P-P_{\text{eq}})\right)^2 + \cdots. \tag{3.14}$$

For the term inside the first integral that appears multiplying $(P-P_{\text{eq}})$, one may use Eq. (3.11) in order to get an arbitrary constant; after integration, using the normalization condition of Eq. (2.9), this first-order term yields zero. For the term inside the first integral that multiplies $(P-P_{\text{eq}})^2$, one may use Eq. (3.5) at equilibrium, in such a way that

$$F(P) - F(P_{\text{eq}}) = \int_{-\infty}^{\infty} dx\ \frac{1}{2}(P-P_{\text{eq}})^2\left.\left\{\frac{\Omega[P]}{\Psi[P]}\right\}\right|_{P=P_{\text{eq}}(x)} - \left.\frac{1}{2\beta}\frac{d^2\Lambda[Q]}{dQ^2}\right|_{P_{\text{eq}}(x)}\left(\int_{-\infty}^{\infty} dx \left.\frac{dg[P]}{dP}\right|_{P_{\text{eq}}(x)}(P-P_{\text{eq}})\right)^2 + \cdots. \tag{3.15}$$

The equation above yields $[F(P)-F(P_{\text{eq}})] \geq 0$ provided that one uses the previously defined properties for the quantities $\Omega[P]$ and $\Psi[P]$, and additionally, one supposes that $(d^2\Lambda[Q]/dQ^2)|_{P_{\text{eq}}(x)} < 0$.

Let us now analyze the situation of an isolated system, i.e., $\phi(x)=$constant; in this case, the H theorem should be expressed in terms of the time derivative of the entropy, in such a way that Eq. (3.4) should be replaced by

$$\frac{dS[P]}{dt} = -\int_{-\infty}^{\infty} dx\left(\Omega[P]\frac{\partial P}{\partial x}\right)\left(\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2}\frac{\partial P}{\partial x}\right) = -\int_{-\infty}^{\infty} dx\,\Omega[P]\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2}\left(\frac{\partial P}{\partial x}\right)^2 \geq 0. \tag{3.16}$$

As expected, the proof of the H theorem for an isolated system becomes much simpler than that for the system in the presence of an external potential. In particular, there is no requirement for a relation involving the parameters of the FPE and the entropy, like the one of Eq. (3.5); all that one needs is a standard condition associated with the FPE, i.e., $\Omega[P]\geq 0$, the restriction $d\Lambda[Q]/dQ\geq 0$ for the outer functional of the entropy, as well as the general restrictions of Eq. (3.1) for the entropy.

IV. SOME FAMILIES OF FPEs AND THEIR ASSOCIATED ENTROPIES

In this section we will explore further the correspondence between the nonlinear FPE of Eq. (2.5) and general entropic forms, established through Eq. (3.5). This equation shows clearly that there may be families of FPEs, corresponding to the same ratio $(\Omega[P]/\Psi[P])$, associated with a single entropic form; i.e., the same entropy may be associated with different dynamical processes. In the following examples, we consider classes of FPEs satisfying

$$\Omega[P] = a[P]b[P], \quad \Psi[P] = a[P]P, \tag{4.1}$$

where the functionals $a[P]$ and $b[P]$ are restricted by the conditions imposed previously for the functionals $\Omega[P]$ and $\Psi[P]$. In addition to that, in the first three examples we will consider entropic forms characterized by $\Lambda(Q[P])=Q[P]$; for these cases Eq. (3.5) becomes

$$\frac{d^2g[P]}{dP^2} = -\beta\frac{b[P]}{P}. \tag{4.2}$$

Therefore one has a freedom for choosing different forms for the functional $a[P]$, leading to the same entropic form. Next, we work out some examples.

(a) *The class of FPEs associated with the Boltzmann-Gibbs entropy*: This class corresponds to the functionals $\Omega[P]$ and $\Psi[P]$ satisfying Eq. (4.1), with $b[P]=D$ (constant). Integrating Eq. (4.2) one gets

$$\frac{dg}{dP} = -\beta D\ln P + C \Rightarrow g[P] = -k_B P\ln P, \tag{4.3}$$

where we have used the conditions $g(0)=g(1)=0$ to eliminate the constant C, and set the Lagrange multiplier $\beta=k_B/D$, where k_B represents the Boltzmann constant. It should be stressed that usually one associates the Boltzmann-Gibbs entropy with the linear FPE, which represents the simplest equation within the present class. Herein we show that, by properly defining the functionals $\Omega[P]$ and $\Psi[P]$, one may get nonlinear FPEs, with time-dependent solutions that may be different from standard exponential probability distributions, but still associated with the Boltzmann-Gibbs entropy. This whole family of FPEs presents the Boltzmann-Gibbs distribution as the stationary-state solution. As a simple example of this class, one may have the nonlinear FPE characterized by $a[P]=P^{\nu}$ $(\nu\in\Re)$ and $b[P]=D$ (constant).

(b) *The class of FPEs associated with Tsallis' entropy*: It is important to notice that the simplest FPE of this class was originally proposed with $\Psi[P(x,t)]=P(x,t)$ and $\Omega[P(x,t)]=(2-q)D[P(x,t)]^{1-q}$, where D is a constant [12], however, it is very common in the literature [24–26] to find this FPE with the replacement $2-q\rightarrow q$. Herein we shall consider this class of FPEs in such a way to satisfy Eq. (4.1), with $b[P(x,t)]=qD[P(x,t)]^{q-1}$; integrating Eq. (4.2),

$$g[P] = -\frac{\beta D}{q-1}P^q + CP \Rightarrow g[P] = k\frac{P-P^q}{q-1}, \tag{4.4}$$

where we have set $\beta=k/D$ (k is a constant with dimensions of entropy) and have also used the conditions $g(0)=g(1)=0$ to eliminate the constant C. In Eq. (4.4) one readily recognizes the entropy proposed by Tsallis [27] that depends on the well-known entropic index q. Similarly to example (a), one has a whole class of FPEs, corresponding to different choices for the functional $a[P]$ of Eq. (4.1), some of which exhibit time-dependent solutions different from the ones presented in Refs. [12,13], but all of them associated with the entropic form of Eq. (4.4). This whole family of FPEs presents the Tsallis distribution (also known as q exponential) [24–26] as the stationary-state solution.

(c) *The class of FPEs associated with the entropy of Refs.* [36,37]. : In this example we proceed in an inverse way with respect to the previous two cases; i.e., we start from a given entropic form, in order to find the class of FPEs associated with it. Let us then consider [36,37]

$$g[P] = k[1-\exp(-cP) + Pg_0], \quad [g_0 = \exp(-c) - 1], \tag{4.5}$$

where c is an arbitrary dimensionless constant, and k is a constant with dimensions of entropy. Substituting into Eq. (4.2), one gets

$$b[P] = -DP[1 - c^2\exp(-cP)], \tag{4.6}$$

where we have set $D=k/\beta$. The functional form above defines the family of FPEs associated with different definitions for the functional $a[P]$, all of them related to the entropic form of Eq. (4.5); the simplest of these equations corresponds to $a[P]=1$.

(d) *The class of FPEs associated with the Renyi entropy* [38]: Similarly to the previous example, we start from the entropic form, in order to find the class of FPEs associated with it. In this case we have that

$$\Lambda(Q[P]) = k\frac{\ln Q[P]}{1-q}, \quad \frac{d\Lambda[Q]}{dQ} = \frac{k}{(1-q)Q[P]}, \quad g[P] = P^q, \tag{4.7}$$

where k is a constant with dimensions of entropy. It is important to stress that in order to satisfy the H theorem, entropic forms characterized by an outer functional $\Lambda[Q]$ are restricted to the condition that $(d\Lambda[Q]/dQ)$ should present a sign different from the one of $(d^2g[P]/dP^2)$ (as assumed in the beginning of Sec. III), as can be seen from simple analyses of Eq. (3.5), for the case of a system in the presence of an external potential, or of Eq. (3.16), for the case of an isolated system. Substituting the functionals of Eq. (4.1) into Eq. (3.5) one gets

$$-\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{d^2g[P]}{dP^2}=\frac{b[P]}{P}, \tag{4.8}$$

and using Eq. (4.7),

$$b[P]=\frac{Dq}{Q[P]}P^{q-1}=\frac{DqP^{q-1}}{\int_{-\infty}^{\infty}dx\ P^q}, \tag{4.9}$$

where we have set $D=k/\beta$. It is important to remember that the functionals $\Omega[P]$ and $\Psi[P]$ are supposed to be both positive, for a well-defined FPE, which implies $a[P],b[P]>0$ [cf. Eq. (4.1)]. From Eq. (4.9) this condition is not satisfied if $q\leq 0$. Notice that this entropic form satisfies the condition $(d^2\Lambda[Q]/dQ^2)_{P_{\mathrm{eq}}(x)}<0$, required by the H theorem [cf. Eq. (3.15)], for $q<1$; therefore, one can assure the validity of such an entropic form, from the physical point of view, for the interval $0<q<1$.

From the entropic forms discussed above, surely the Boltzmann-Gibbs form represents the most common in nature, being applicable for systems characterized by short-range interactions and/or weak correlations. For these systems, such an entropy is extensive, as required by thermodynamics. However, for strongly correlated systems, one may have a Boltzmann-Gibbs entropy that turns out to be nonextensive, and so, in order to match satisfactorily thermodynamics, one should look for other types of entropic forms, like the ones defined above, in such a way to recover extensivity. From these, the one proposed by Tsallis appears as the most found in natural systems; as a typical example, one could mention quantum spin chains, for which Tsallis entropy is extensive for a value of $q\neq 1$ [31,32]. Besides that, several examples in the literature suggest that other entropic forms (distinct from the Boltzmann-Gibbs one) should be applicable; this is usually indicated by an analysis of the corresponding distribution functions, which are associated to given entropic forms, through standard entropy-maximization procedures. Many systems exhibit some kind of distribution that has been associated with Tsallis distribution; it is important to mention that any entropic form given by a monotonically increasing functional of Tsallis entropy leads to the same distribution, under maximization. As an example, Tsallis' and Renyi's entropies share the same distribution; however, as we have seen above, this later entropy is well defined only for the interval $0<q<1$, leading to a significant restriction for its applicability. As concrete examples of possible applications for Tsallis entropy, we mention (i) self-gravitating systems, characterized by $q<7/9$ [5,39]; (ii) relaxation of two-dimensional turbulence in a pure-electron plasma, for which $q=2$ [7]; (iii) anomalous diffusion in an optical lattice (in this case, the value of q differs from 1 by an amount that depends on the ratio of typical energy parameters of the system, like the recoil energy and the potential depth) [8]; (iv) seismic time series ($q\approx 2.98$) [40]; and (v) distribution of daily returns of financial stock markets (e.g., SP100 ($q=1.4$) [41] and DJ30 ($q\approx 1.45$) [42]). Another type of one-parameter entropic form (whose corresponding nonlinear FPE appears as a particular case of Eq. (2.5) and has already been discussed in detail elsewhere [23]) seems to be appropriate for a description of some relativistic phenomena [9]; in this case, the dimensionless entropic parameter is associated to the ratio of velocities of special relativity (v/c). Further discussions on applications of generalized entropies for stellar dynamics, two-dimensional turbulence, and bacterial populations may also be found in Refs. [43,44].

V. FPE FOR A MORE GENERAL FREE-ENERGY FUNCTIONAL

In this section we will consider a slightly different FPE, with respect to the one of Eq. (2.5), namely,

$$\frac{\partial P(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(\Psi[P]\frac{\partial}{\partial x}(\phi(x)\chi[P])\right)+\frac{\partial}{\partial x}\left(\Omega[P]\frac{\partial P}{\partial x}\right), \tag{5.1}$$

where a new functional $\chi[P]$ was introduced [notice that Eq. (2.5) is recovered for $\chi[P]=1$] that should be finite and positive definite. The interesting point about such a FPE is that it is consistent with the definition of a "generalized internal energy" [24–26],

$$U=\int_{-\infty}^{\infty}dx\ \phi(x)\Gamma[P(x,t)], \tag{5.2}$$

where we are assuming that $\Gamma[P]$ represents a positive, monotonically increasing functional of $P(x,t)$, that should be at least $\Gamma[P]\in C^1$.

Now, we take this internal energy in the free-energy functional of Eq. (3.2) and consider the same entropic form of Eq. (3.1). Let us then prove the H theorem for such a system, following the same steps of Sec. III; one gets that

$$\begin{aligned}\frac{dF}{dt}&=\frac{d}{dt}\left(\int_{-\infty}^{\infty}dx\ \phi(x)\Gamma[P(x,t)]-\frac{1}{\beta}\Lambda(Q[P])\right)\\&=\int_{-\infty}^{\infty}dx\left(\phi(x)\frac{d\Gamma[P]}{dP}-\frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right)\frac{\partial P}{\partial t}.\end{aligned} \tag{5.3}$$

Using the FPE of Eq. (5.1) and integrating by parts, one obtains

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \left\{ \Psi[P]\frac{\partial}{\partial x}(\phi(x)\chi[P]) + \Omega[P]\frac{\partial P}{\partial x} \right\} \times \left\{ \frac{\partial}{\partial x}\left(\phi(x)\frac{d\Gamma[P]}{dP} \right) - \frac{1}{\beta}\frac{d\Lambda[Q]}{dQ}\frac{d^2 g[P]}{dP^2}\frac{\partial P}{\partial x} \right\}. \tag{5.4}$$

The *H* theorem applies; i.e.,

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx\, \Psi[P]\left\{ \frac{\partial}{\partial x}(\phi(x)\chi[P]) + \frac{\Omega[P]}{\Psi[P]}\frac{\partial P}{\partial x} \right\}^2 \leq 0, \tag{5.5}$$

provided that Eq. (3.5) holds, with an additional restriction for the functional $\chi[P]$,

$$\chi[P] = \frac{d\Gamma[P]}{dP}. \tag{5.6}$$

It should be mentioned that the constraint above, relating the functional $\chi[P]$ of the FPE with the quantity $\Gamma[P]$ that appears in the definition of the generalized internal energy (with $\Gamma[P] \neq P$) has to be introduced, in such a way to satisfy the *H* theorem.

Let us now show that, at equilibrium, the condition of Eq. (3.5) is equivalent to the maximum-entropy principle, when one uses the FPE of Eq. (5.1). Defining the functional

$$\mathcal{I}[P(x,t)] = \Lambda(Q[P]) + \alpha\left(1 - \int_{-\infty}^{\infty} dx\, P(x,t) \right) + \beta\left(U - \int_{-\infty}^{\infty} dx\, \phi(x)\Gamma[P(x,t)] \right) \tag{5.7}$$

(α and β are Lagrange multipliers) one has that

$$\left.\frac{d\mathcal{I}[P]}{dP}\right|_{P=P_{\mathrm{eq}}(x)} = 0 \Rightarrow \left.\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right|_{P=P_{\mathrm{eq}}(x)} - \alpha - \beta\phi(x)\left.\frac{d\Gamma[P]}{dP}\right|_{P=P_{\mathrm{eq}}(x)} = 0, \tag{5.8}$$

where $P_{\mathrm{eq}}(x)$ represents the probability distribution at equilibrium. Considering Eq. (5.1) at equilibrium one gets

$$-\frac{\partial}{\partial x}(\phi(x)\chi[P_{\mathrm{eq}}]) = \frac{\Omega[P_{\mathrm{eq}}]}{\Psi[P_{\mathrm{eq}}]}\frac{dP_{\mathrm{eq}}(x)}{dx}, \tag{5.9}$$

and after integration,

$$-\phi(x)\chi[P_{\mathrm{eq}}(x)] + C = \int_{x_0}^{x} dx \frac{\Omega[P_{\mathrm{eq}}]}{\Psi[P_{\mathrm{eq}}]}\frac{dP_{\mathrm{eq}}(x)}{dx} = \int_{P_{\mathrm{eq}}(x_0)}^{P_{\mathrm{eq}}(x)} \frac{\Omega[P_{\mathrm{eq}}(x')]}{\Psi[P_{\mathrm{eq}}(x')]} dP_{\mathrm{eq}}(x'), \tag{5.10}$$

where $C \equiv \phi(x_0)\chi[P_{\mathrm{eq}}(x_0)]$ is a constant. Integrating Eq. (3.5), at equilibrium, and using the equation above, one gets

$$\frac{1}{\beta}\left.\frac{d\Lambda[Q]}{dQ}\frac{dg[P]}{dP}\right|_{P_{\mathrm{eq}}(x)} = \phi(x)\chi[P_{\mathrm{eq}}(x)] + C' = \phi(x)\left.\frac{d\Gamma[P]}{dP}\right|_{P=P_{\mathrm{eq}}(x)} + C', \tag{5.11}$$

where we have used Eq. (5.6) and C' represents another integration constant. The equation above is equivalent to Eq. (5.8), obtained from the maximum-entropy principle.

Therefore, in what concerns the *H* theorem, the necessary relation involving quantities of the FPE with a general entropic form and its equivalence with the maximum-entropy principle, the FPE of Eq. (5.1) is consistent with the definition of a generalized internal energy that is sometimes used in the context of nonextensive statistical mechanics [24–26].

VI. CONCLUSIONS

We have proved the *H* theorem by using general nonlinear Fokker-Planck equations. In order to prove the *H* theorem for a system in the presence of an external potential, a relation involving terms of the Fokker-Planck equation and the entropy of the system was proposed. In principle, one may have classes of Fokker-Planck equations related to a single entropic form. Since the Fokker-Planck equation is a phenomenological equation that specifies the dynamical evolution associated with a given physical system, this relation may be useful in the identification of the entropic form associated with such a system. In particular, the present approach makes it possible to identify entropic forms associated with some anomalous systems, exhibiting unusual behavior, that are known to be appropriately described by nonlinear Fokker-Planck equations, like the ones considered herein. By considering a modified Fokker-Planck equation, we have also proved the *H* theorem for a type of generalized internal energy, like the one used within the nonextensive statistical-mechanics formalism. For that, the same relation connecting the parameters of the Fokker-Planck equation and the corresponding entropic form had to be introduced. To our knowledge, it is the first time that the *H* theorem has been verified, for a system in the presence of an external potential, by considering a nonlinear weight in the definition of the internal energy. Making use of the relation mentioned, we have calculated well-known entropic forms, associated with given Fokker-Planck equations. In the case of the standard Boltzmann-Gibbs entropy, apart from the simplest linear Fokker-Planck equation, one may have a whole class of nonlinear Fokker-Planck equations, whose time-dependent probability distributions may be distinct from simple exponential distributions, but all of them are related to this particular entropic form; the stationary-state solution is the same as the one of the linear Fokker-Planck equation, i.e., a Boltzmann-Gibbs distribution. A similar behavior is verified for more general, nonadditive, entropic forms, e.g., the Tsallis' entropy. Although this relation involving families of

Fokker-Planck equations and entropic forms may not be unique, we have shown that, when considered at equilibrium, it is equivalent to the principle of maximum entropy. The present results suggest that behind such a relation there may be a deep physical insight that deserves further investigations.

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